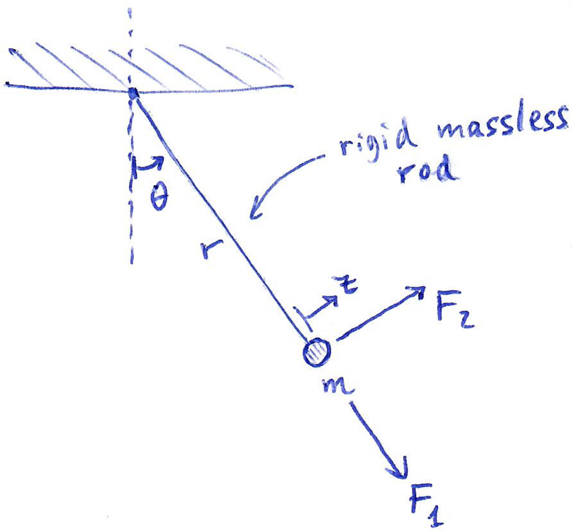
ME 4555 - Lecture 3 - Mechanical systems II

Today: mechanical systems with rotational motion (rotation about an axis)

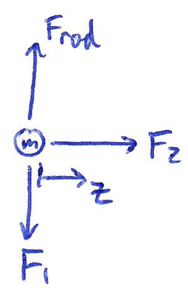
Newton's 2nd law for translational motion ($F = m\ddot{x}$) can be manipulated to characterize rotational motion as well.



F_1 (parallel to rod) is balanced by the rod, which prevents motion in that direction.*

F_2 (perpendicular to rod) causes mass to move instantaneously in the direction of z.

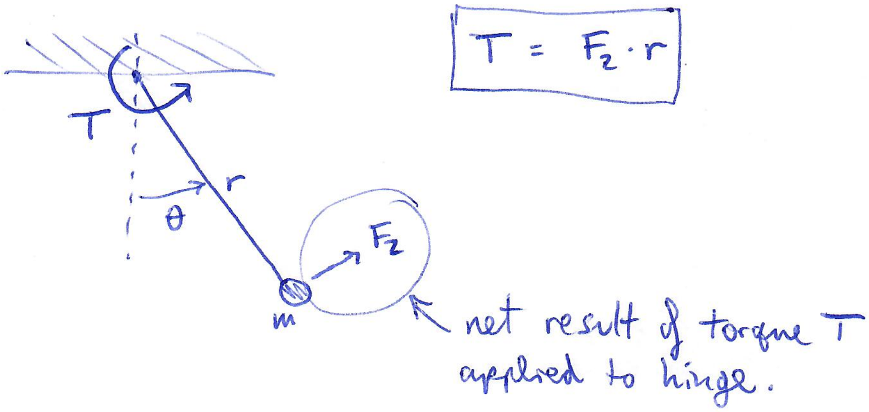
FBD:



$$\left\{ \begin{array}{l} r\ddot{\theta} = \ddot{z} \\ m\ddot{z} = F_2 \end{array} \right\} \Rightarrow \boxed{mr\ddot{\theta} = F_2}$$

Such perpendicular force is often written as a Torque

$$\boxed{T = F_2 \cdot r}$$

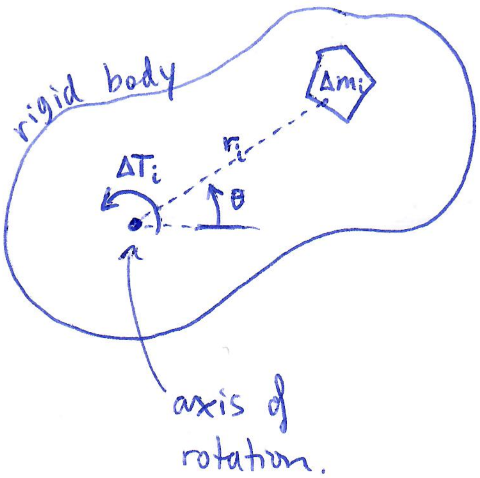


so EOM in terms of θ, T:

$$\boxed{mr^2\ddot{\theta} = T}$$

* It's more complicated than this, but let's ignore it for now since we

A rigid body rotating about an axis can be viewed as a collection of smaller masses, each rotating at the same angular rate about the axis.



From prev. page:

$$\Delta T_i = r_i^2 \Delta m_i \ddot{\theta}$$

Summing over all i (each mass piece):

$$\underbrace{\sum_i \Delta T_i}_{\substack{\text{total torque} \\ \text{at axis of} \\ \text{rotation}}} = \underbrace{\left(\sum_i r_i^2 \Delta m_i \right)}_{\substack{\text{moment of} \\ \text{inertia.} \\ (J)}} \ddot{\theta}$$

$\ddot{\theta}$: angular acceleration

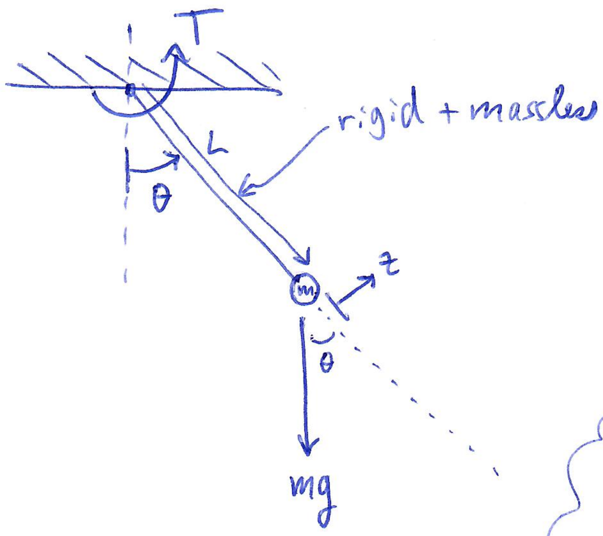
In the limit as Δm_i become infinitesimal, $J = \int r^2 \, dm$

Newton's 2nd law for rotational motion:

$$\underbrace{\sum T}_{\substack{\text{sum of external} \\ \text{torques}}} = \underbrace{J}_{\substack{\text{moment of} \\ \text{inertia of object} \\ \text{about axis of rotation}}} \ddot{\theta}$$

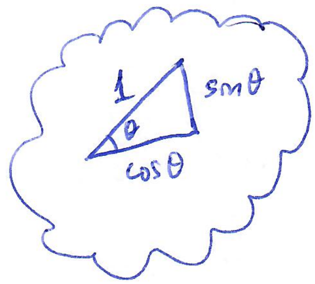
angular acceleration $\left(\ddot{\theta} = \dfrac{d^2\theta}{dt^2} \right)$

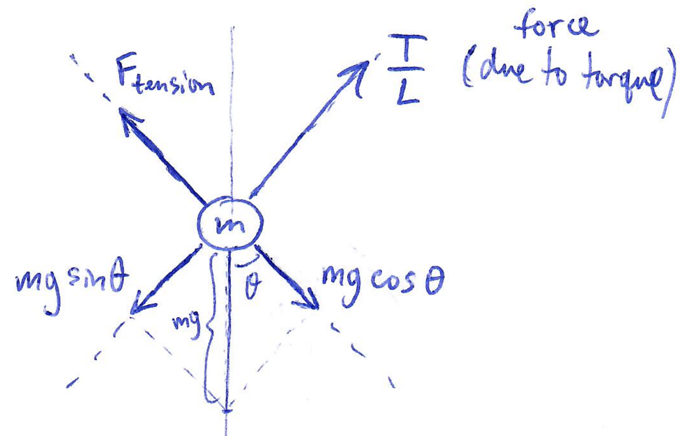
Ex: simple pendulum with gravity.



decompose external forces into parallel + perpendicular components.

FBD:





Let's ignore the parallel direction because we are only concerned with motion in the perpendicular direction.

In the perpendicular direction: $\underbrace{Lm\ddot{\theta}}_{\ddot{z}} = -mg\sin\theta + \frac{T}{L}$

Rearranging: $\boxed{\ddot{\theta} + \frac{g}{L}\sin\theta = \frac{T}{mL^2}}$

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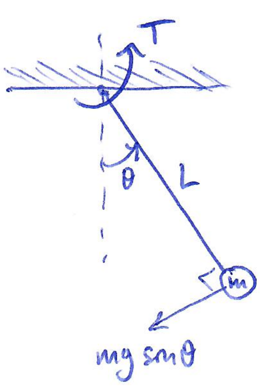
different derivation: use rotational version!



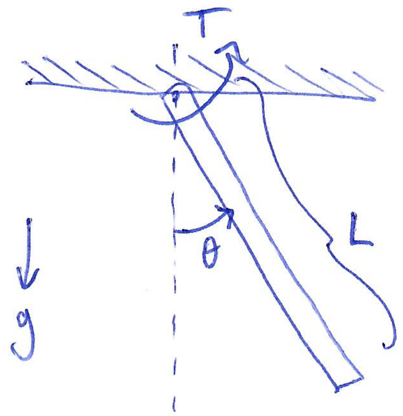
for a mass m distance L from axis, $J = mL^2$.

$$J\ddot{\theta} = \underbrace{T}_{\text{external applied torque}} - \underbrace{Lmg\sin\theta}_{\text{torque due to gravity}}$$

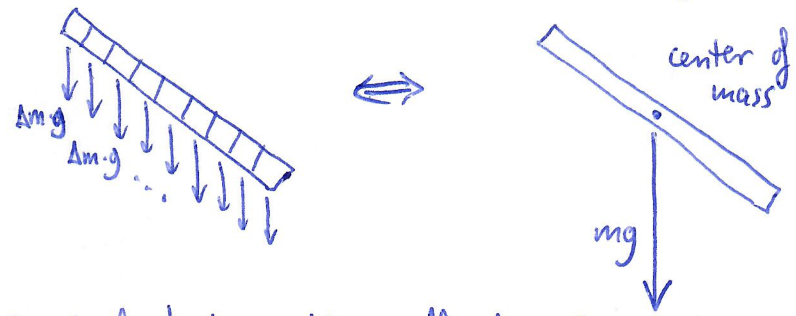
same answer!

$$\Longrightarrow \boxed{mL^2\ddot{\theta} + mgL\sin\theta = T}$$

<u>Ex</u>: Solid pendulum



gravity acts on entire mass of arm



equivalent to putting all the force at the center of mass instead.

In other words: $\sum_i \left(\begin{matrix}\text{torque due to gravity}\\ \text{on segment } i \text{ of the arm}\end{matrix}\right) = (\text{distance to c.m.}) \cdot \left(\begin{matrix}\text{total}\\ \text{gravity}\\ \text{force}\end{matrix}\right)$

Assuming c.m. is $\frac{1}{2}$ way down ($\frac{L}{2}$ from hinge), we have:

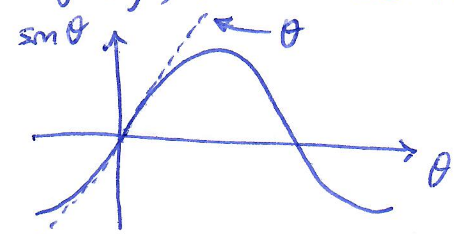
$$J\ddot{\theta} = T - \left(\tfrac{1}{2}L\right)(mg\sin\theta)$$

$$\Rightarrow \boxed{J\ddot{\theta} + \tfrac{1}{2}mgL\sin\theta = T}$$

This equation is <u>nonlinear</u> (depends on $\sin\theta$ instead of just θ).

If θ is close to zero (small angle), then use Taylor approximation

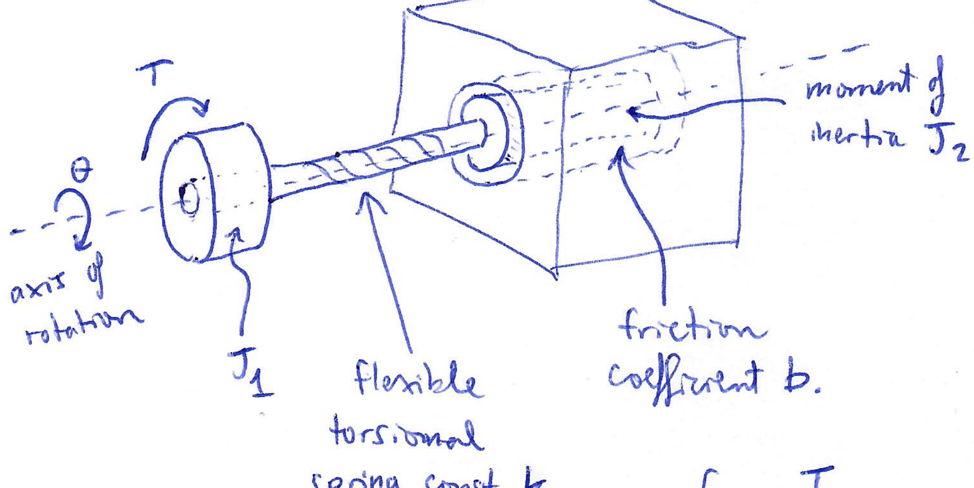
$$\sin\theta \approx \theta \underbrace{- \frac{\theta^3}{6} + \frac{\theta^5}{120} - \ldots}_{\text{higher order terms}}$$



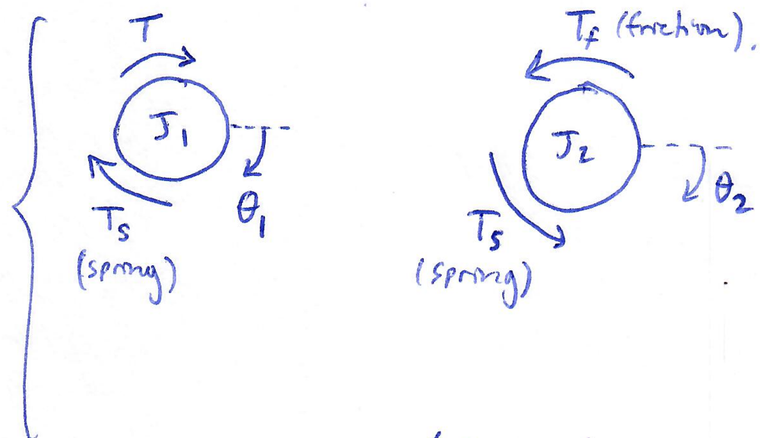
Then a linear approximation to the pendulum dynamics is:

$$\boxed{J\ddot{\theta} + \tfrac{1}{2}mgL\theta = T}$$

Ex: torsional spring-damper system



view from the left FBD:

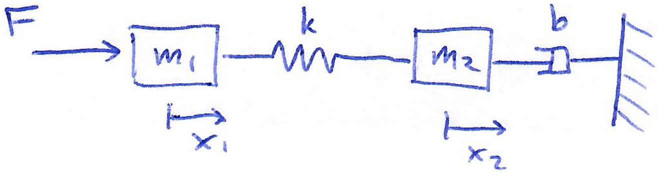


First mass: $$J_1 \ddot{\theta}_1 = T + \underbrace{k(\theta_2 - \theta_1)}_{T_s}$$ (spring torque caused by difference in angles)

Second mass: $$J_2 \ddot{\theta}_2 = \underbrace{-k(\theta_2 - \theta_1)}_{T_s} \underbrace{- b\dot{\theta}_2}_{T_f}$$

EOM solution:
$$\begin{cases} J_1 \ddot{\theta}_1 + k\theta_1 - k\theta_2 = T \\ J_2 \ddot{\theta}_2 + k\theta_2 - k\theta_1 + b\dot{\theta}_2 = 0 \end{cases}$$

Analogous linear translational system:



if we let $m_i \leftrightarrow J_i$

$F \leftrightarrow T$

$x_i \leftrightarrow \theta_i$

the EOM are the same!